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Some inequalities on generalized Schur complements

Bo-Ying Wang [a,1], Xiuping Zhang [a], Fuzhen Zhang [b,*,2]

[a] *Department of Mathematics, Beijing Normal University, Beijing 100875, People's Republic of China*

[b] *Department of Mathematical Sciences, Nova Southeastern University, 3301 College Avenue, Fort Lauderdale, FL 33314, USA*

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Abstract

This paper presents some inequalities on generalized Schur complements. Let A be an $n \times n$ (Hermitian) positive semidefinite matrix. Denote by A/α the generalized Schur complement of a principal submatrix indexed by a set α in A. Let A^+ be the Moore–Penrose inverse of A and $\lambda(A)$ be the eigenvalue vector of A. The main results of this paper are:

1. $\lambda(A^+(\alpha')) \geqslant \lambda((A/\alpha)^+)$, where α' is the complement of α in $\{1, 2, \ldots, n\}$.
2. $\lambda(A^r/\alpha) \leqslant \lambda^r(A/\alpha)$ for any real number $r \geqslant 1$.
3. $(C^*AC)/\alpha \leqslant C^*/\alpha \; A(\alpha') \; C/\alpha$ for any matrix C of certain properties on partitioning.

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* Corresponding author. Tel.: +1-954-262-8317; fax: +1-954-262-3931; e-mail: zhang@polaris.nova.edu

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1. Introduction

The Schur complements have appeared to be a useful tool in the study of matrix theory and its applications to statistics. A great deal of work on the topic has been done by a number of authors. Readers are referred to Haynsworth [7–9] for initial work on the subject, Carlson et al. [6] on the Schur complements in terms of Moore–Penrose inverse (generalized Schur complements), Cottle [5] for manifestations of Schur complements, Ando [2] on generalized Schur complements, Lyapunov stability, Styan [11] on Schur complements and linear statistical models, Bapat [3] using Schur complements to refine the Oppenheim's inequality, Butler and Morley [4] on the equivalence of six generalized Schur complements, Smith [12] on interlacing eigenvalues of Schur complements, and econometrics, and Wang and Zhang [14] on Schur complements and Hadamard products.

Considerable interest in recent work on (generalized) Schur complements has been witnessed. See the references for extensive studies and applications in the area.

The purpose of this paper is to present some new matrix (eigenvalue) inequalities on generalized Schur complements of positive semidefinite matrices.

To begin with, let α and β be proper index subsets of $\{1, 2, \ldots, n\}$. For any $n \times n$ complex matrix A, denote by $A(\alpha, \beta)$, or simply $A(\alpha)$ if $\alpha = \beta$, the submatrix of A lying in rows α and columns β. Let α' be the complement of α in $\{1, 2, \ldots, n\}$. Evidently, there exists a permutation matrix P such that

$$P^{\mathrm{T}} A P = \begin{pmatrix} A(\alpha) & A(\alpha, \alpha') \\ A(\alpha', \alpha) & A(\alpha') \end{pmatrix},$$

where P^{T} is the transpose of P. We may assume that A is the block matrix; the results carry over to the general case via permutations.

The *generalized Schur complement* of the principal submatrix $A(\alpha)$ in A is defined and denoted as follows:

$$A/\alpha = A(\alpha') - A(\alpha', \alpha)(A(\alpha))^{+} A(\alpha, \alpha'). \tag{1}$$

where $(A(\alpha))^{+}$ is the Moore–Penrose inverse of matrix $A(\alpha)$.

The Schur complements (in case $A(\alpha)$ is nonsingular) and generalized Schur complements, studied by a number of authors, have applications in statistics, applied mathematics and other fields (see the references). It can be shown that the Moore–Penrose inverse $(A(\alpha))^{+}$ in (1) may be replaced by any $\{1\}$-generalized inverse $(A(\alpha))^{(1)}$ when A is positive semidefinite; in other words, A/α is independent of choice of the $\{1\}$-generalized inverse in the definition when A is positive semidefinite.

We shall use two well-known facts: First (see, e.g., [1] or [16, p. 184]), if a partitioned matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is positive semidefinite, where A and D are square, then

$$B = AR \tag{2}$$

for some matrix R, and

$$C = CA^{(1)}A \tag{3}$$

for every $\{1\}$-generalized inverse $A^{(1)}$ of A.

The identity (2) is equivalent to $\mathscr{R}(B) \subseteq \mathscr{R}(A)$; that is, the column space of B is contained in that of A. And also (3) is equivalent to $\mathscr{R}(C^*) \subseteq \mathscr{R}(A^*)$, where $*$ means conjugate transpose.

It is easy to see, with this fact, that if A is an n-square positive semidefinite matrix, then A is transformed into $A(\alpha) \oplus (A/\alpha)$ by congruence. In symbols, with k standing for the number of the elements contained in α,

$$CAC^* = \begin{pmatrix} A(\alpha) & 0 \\ 0 & A/\alpha \end{pmatrix},$$

where

$$C = \begin{pmatrix} I_k & 0 \\ -A(\alpha', \alpha)(A(\alpha))^+ & I_{n-k} \end{pmatrix}. \tag{4}$$

It follows that if A is positive semidefinite, then so is A/α.

Second (see, e.g., [10, p. 18] or [15, p. 54]), if A is invertible, then A^{-1} takes the form

$$A^{-1} = \begin{pmatrix} (A/\alpha')^{-1} & * \\ * & (A/\alpha)^{-1} \end{pmatrix},$$

where $*$ denotes entries irrelevant to our discussions. It follows that

$$A^{-1}(\alpha') = (A/\alpha)^{-1}. \tag{5}$$

With this identity one may convert many results on principal submatrices into Schur complements; vice versa. This idea does not work for generalized Schur complements, since $A^+(\alpha') = (A/\alpha)^+$ does not hold in general as we shall see in the later examples. Moreover, as is well known, for any positive definite A,

$$A^{-1}(\alpha) \geqslant (A(\alpha))^{-1}.$$

This does not generalize to the Moore–Penrose inverse; that is, it is not true in general that

$$A^{+}(\alpha) \geqslant (A(\alpha))^{+}.$$

Thus the generalized Schur complement is not simply an extension of the Schur complement.

This paper deals with, using a continuity argument, the inequalities of the generalized Schur complements of positive semidefinite matrices.

2. Eigenvalue inequalities of Schur complements

As usual, we write $A > 0$ (or $\geqslant 0$) if A is positive (semi-)definite, and $A \leqslant B$ or $B \geqslant A$ if $B - A \geqslant 0$ for Hermitian matrices A and B of the same size. Obviously $A \geqslant 0 \Rightarrow A^{+} \geqslant 0$.

We assume $A \geqslant 0$ from now on. Denote by $\lambda(A)$ the row vector of the eigenvalues of A arranged in decreasing order, and by $\lambda^{r}(A)$ the row vector of components of $\lambda(A)$ raised to power r. If a vector x is dominated by a vector y entrywise, we write $x \leqslant y$.

If A is invertible, then $A^{+}(\alpha') = (A/\alpha)^{+}$ by (5). If A is singular, these two matrices may not be comparable. Take $\alpha = \{1\}$ and

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Then

$$A^{+} = \frac{1}{9}\begin{pmatrix} 5 & 1 & -4 \\ 1 & 2 & 1 \\ -4 & 1 & 5 \end{pmatrix}$$

and

$$A^{+}(\alpha') = \frac{1}{9}\begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}, \quad (A/\alpha)^{+} = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

It is readily seen that neither $A^{+}(\alpha') \leqslant (A/\alpha)^{+}$ nor $A^{+}(\alpha') \geqslant (A/\alpha)^{+}$ holds. The eigenvalues, however, can be compared and are related in the following inequality; and besides, $A^{+}(\alpha') = (A/\alpha)^{(1)}$ for some $\{1\}$-generalized inverse, as we shall see in the proof. (This example also shows that the inequality $A^{+}(\alpha) \geqslant (A(\alpha))^{+}$, as we mentioned at the end of Section 1, does not hold in general.)

Theorem 1. *Let $A \geqslant 0$. Then*

$$\lambda(A^{+}(\alpha')) \geqslant \lambda((A/\alpha)^{+}). \tag{6}$$

Proof. Upon computation for the lower right block of CAC^* in (4), we have

$$(-A(\alpha', \alpha)(A(\alpha))^+, I_{n-k})A(-A(\alpha', \alpha)(A(\alpha))^+, I_{n-k})^* = A/\alpha.$$

With $AA^+A = A$, we have

$$\begin{aligned} A/\alpha &= (-A(\alpha', \alpha)(A(\alpha))^+, I_{n-k})AA^+A(-A(\alpha', \alpha)(A(\alpha))^+, I_{n-k})^* \\ &= (0, A/\alpha)A^+(0, A/\alpha)^* \\ &= A/\alpha\, A^+(\alpha')\, A/\alpha. \end{aligned}$$

That is, $A^+(\alpha')$ is a $\{1\}$-generalized inverse of A/α:

$$A/\alpha\, A^+(\alpha')\, A/\alpha = A/\alpha. \tag{7}$$

Notice that $A/\alpha \geqslant 0$. Let U be a unitary matrix such that

$$A/\alpha = U^* \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} U,$$

where $D > 0$ is a diagonal matrix. Then

$$(A/\alpha)^+ = U^* \begin{pmatrix} D^{-1} & 0 \\ 0 & 0 \end{pmatrix} U.$$

On the other hand, by (7) and the property of $\{1\}$-generalized inverses, we have

$$A^+(\alpha') = U^* \begin{pmatrix} D^{-1} & C \\ C^* & E \end{pmatrix} U \geqslant 0$$

for some matrices C and E. It follows, by the eigenvalue interlacing theorem on principal submatrices ([10, p. 185]) that $\lambda(A^+(\alpha')) \geqslant \lambda((A/\alpha)^+)$. $\square$

Note that identity (7) saying that $A^+(\alpha')$ is some $\{1\}$-generalized inverse of A/α, generalizes (5) for positive definite matrices.

We now turn our attention to matrix powers. It is known from [13] that for $A \geqslant 0$ and $r \geqslant 1$

$$\lambda(A^r(\alpha)) \geqslant \lambda^r(A(\alpha)). \tag{8}$$

For the analog of Schur complements we have the following theorem.

Theorem 2. *Let $A \geqslant 0$. Then for any number $r \geqslant 1$*

$$\lambda(A^r/\alpha) \leqslant \lambda^r(A/\alpha). \tag{9}$$

Proof. If A is nonsingular, the assertion follows, since, by (5),

$$A/\alpha = (A^{-1}(\alpha'))^{-1}.$$

An application of (8) then gives (9) for the nonsingular case.

Suppose that A is singular. We first show that the limit representation for the Schur complement A/α:

$$\lim_{\varepsilon\to 0+}(A+\varepsilon I)/\alpha = A/\alpha. \tag{10}$$

Let U be a unitary matrix such that

$$A(\alpha) = U^*\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}U,$$

where $D > 0$ is a diagonal matrix. Then for any $\varepsilon > 0$, by computation and (2),

$$\begin{aligned}(A+\varepsilon I)/\alpha &= (A(\alpha')+\varepsilon I) - A(\alpha',\alpha)((A+\varepsilon I)(\alpha))^{-1}A(\alpha,\alpha') \\ &= (A(\alpha')+\varepsilon I) - R^*A(\alpha)((A+\varepsilon I)(\alpha))^{-1}A(\alpha)R \\ &= (A(\alpha')+\varepsilon I) - R^*U^*\begin{pmatrix} D(D+\varepsilon I)^{-1}D & 0 \\ 0 & 0 \end{pmatrix}UR \\ &\to A(\alpha') - R^*A(\alpha)R \ (\text{as } \varepsilon\to 0) \\ &= A(\alpha') - R^*A(\alpha)(A(\alpha))^+A(\alpha)R \\ &= A(\alpha') - A(\alpha',\alpha)(A(\alpha))^+A(\alpha,\alpha') \\ &= A/\alpha.\end{aligned}$$

Notice that $A \geqslant B > 0 \ \Rightarrow A/\alpha \geqslant B/\alpha$. This can be extended to the singular case by the above representation.

Since $A+\varepsilon I > 0$ for every $\varepsilon > 0$, we have, by the nonsingular case,

$$\lambda(A^r/\alpha) \leqslant \lambda((A+\varepsilon I)^r/\alpha) \leqslant \lambda^r((A+\varepsilon I)/\alpha).$$

Letting $\varepsilon \to 0$ implies the desired inequality. $\square$

Remark 1. It can be shown for $A \geqslant 0$ and $r \geqslant 1$ that

$$A/\alpha \geqslant (A^r/\alpha)^{1/r}$$

and that, by the eigenvalue interlacing theorem on principal submatrices again,

$$\lambda_t(A) \geqslant \lambda_t(A^{1/r}/\alpha)^r \geqslant \lambda_t(A/\alpha) \geqslant \lambda_t((A^r/\alpha)^{1/r}) \geqslant \lambda_{t+k}(A) \tag{11}$$

for each $t = 1, \ldots, n-k$, where k is the number of elements contained in α.

Remark 2. The limit representation (10) is not true in general if A is not positive semidefinite. Take $\alpha = \{1, 2\}$ and

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

Then

$$2 = \lim_{\varepsilon \to 0+} (A + \varepsilon I)/\alpha \neq A/\alpha = 1.$$

Remark 3. With the representation (10), many results on Schur complements can be extended to the generalized Schur complements. For instance

$$A, B \geqslant 0 \Rightarrow (A \circ B)/\alpha \geqslant (A/\alpha) \circ (B/\alpha),$$

where $\circ$ stands for the Hadamard product or the entrywise product.

In addition, if A, B, C and D are matrices of appropriate sizes such that $A \geqslant 0$, $B \geqslant 0$, $\mathscr{R}(C) \subseteq \mathscr{R}(A)$ and $\mathscr{R}(D) \subseteq \mathscr{R}(B)$, then

$$\begin{pmatrix} A & C \\ C^* & C^*A^+C \end{pmatrix} \geqslant 0, \quad \begin{pmatrix} B & D \\ D^* & D^*B^+D \end{pmatrix} \geqslant 0,$$

and (by Schur theorem on Hadamard product of positive semidefinite matrices)

$$\begin{pmatrix} A \circ B & C \circ D \\ C^* \circ D^* & (C^*A^+C) \circ (D^*B^+D) \end{pmatrix} \geqslant 0.$$

It follows by taking the Schur complement that

$$(C \circ D)^*(A \circ B)^+(C \circ D) \leqslant (C^*A^+C) \circ (D^*B^+D).$$

These two results have appeared in [14] for the case $A, B > 0$.

Note that, unlike the nonsingular case, $(A \circ B)^+ \leqslant A^+ \circ B^+$ is not true in general.

3. Ordinary products and Schur complements

It is easy to construct an example that A^3/α and $(A/\alpha)^3$ are not comparable for an $A > 0$. Thus in general for $A, B > 0$, $(BAB)/\alpha$ and $B/\alpha\, A/\alpha\, B/\alpha$ are not comparable. We have, however, for $A > 0$ and $B \geqslant 0$ of the same size

$$(BA^{-1}B)/\alpha \leqslant B/\alpha\, (A/\alpha)^{-1}\, B/\alpha. \tag{12}$$

A more general result is the following.

Theorem 3. *Let $A \geqslant 0$ and let C be an $n \times n$ complex matrix. If*

$$\mathscr{R}(C(\alpha, \alpha')) \subseteq \mathscr{R}(C(\alpha)), \tag{13}$$

then

$$(C^*AC)/\alpha \leqslant C^*/\alpha\, A(\alpha')\, C/\alpha. \tag{14}$$

In particular

$$(C^*C)/\alpha \leqslant C^*/\alpha\, C/\alpha. \tag{15}$$

Proof. We begin with two observations which can be verified by straightforward computations:

$$(C/\alpha)^* = C^*/\alpha, \quad \text{for any } C \tag{16}$$

and

$$(Q^*AQ)/\alpha = A/\alpha \quad \text{for } A \geqslant 0, \tag{17}$$

where, for any $k \times (n-k)$ matrix Y,

$$Q = \begin{pmatrix} I_k & Y \\ 0 & I_{n-k} \end{pmatrix}.$$

Now noticing that, by (3),

$$\mathscr{R}(C(\alpha,\alpha')) \subseteq \mathscr{R}(C(\alpha)) \Longleftrightarrow C(\alpha,\alpha') = C(\alpha)(C(\alpha))^+C(\alpha,\alpha'),$$

we may write

$$C = \begin{pmatrix} C(\alpha) & 0 \\ C(\alpha',\alpha) & C/\alpha \end{pmatrix}\begin{pmatrix} I_k & (C(\alpha))^+C(\alpha,\alpha') \\ 0 & I_{n-k} \end{pmatrix}.$$

Let

$$K = \begin{pmatrix} C(\alpha) & 0 \\ C(\alpha',\alpha) & C/\alpha \end{pmatrix}, \quad Q = \begin{pmatrix} I_k & (C(\alpha))^+C(\alpha,\alpha') \\ 0 & I_{n-k} \end{pmatrix}.$$

Then

$$\begin{aligned} (C^*AC)/\alpha &= (Q^*K^*AKQ)/\alpha \\ &= \left(Q^*\begin{pmatrix} * & * \\ * & (C/\alpha)^*A(\alpha')C/\alpha \end{pmatrix}Q\right)\Big/\alpha \\ &= (Q^*GQ)/\alpha \ \ (\text{where } G = K^*AK \geqslant 0) \\ &= G/\alpha \ (\text{by } (17)) \\ &\leqslant G(\alpha') \\ &= (C/\alpha)^*A(\alpha')C/\alpha \\ &= C^*/\alpha\, A(\alpha')\, C/\alpha \ \ (\text{by } (16)). \quad \square \end{aligned}$$

Note that if $C \geqslant 0$ or $C(\alpha)$ is invertible, then (13) is automatically satisfied. Thus (12) follows from (14) due to (5). In addition, one can obtain, assuming that A, $B \geqslant 0$ and that $C(\alpha)$ is invertible,

$$(BAB)/\alpha \leqslant B/\alpha\, A(\alpha')\, B/\alpha$$

and

$$(C^*A^{-1}C)/\alpha \leqslant C^*/\alpha\, (A/\alpha)^{-1}\, C/\alpha.$$

Remark 4. The inequality $(C^*C)/\alpha \leqslant C^*/\alpha\, C/\alpha$ does not hold in general. Take $\alpha = \{1, 2\}$ and

$$C = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$1 = (C^*C)/\alpha \nleqslant C^*/\alpha\, C/\alpha = 0.$$

Remark 5. The condition (13) cannot be replaced by $\mathscr{R}(C^*(\alpha, \alpha')) \subseteq \mathscr{R}(C^*(\alpha))$, as the above example shows.

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